Symplectic Decoupling in n Dimensions

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Contents

Hamilton function with symmetric $2\,n \times 2\,n$-matrix $\mathbf{A}$:

$$\mathcal{H} = \frac{1}{2}\,\psi^T\,\mathbf{A}\,\psi\,,$$

where $\psi = (q_1, p_1, \dots, q_n, p_n)^T$. Equations of Motion (EQOM):

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ \dot{\psi} &= \gamma_0\,\nabla_\psi\,\mathcal{H} \\ &= \gamma_0\,\mathbf{A}\,\psi = \mathbf{F}\,\psi\,, \end{aligned}$$

where the symplectic unit matrix γ_0 has the structure:

$$\gamma_0 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \dots \\ & & & \dots & \end{pmatrix}$$

Decoupled Systems

If the matrix **A** is diagonal, then the Hamiltonian is decoupled:

$$\mathcal{H} = \frac{1}{2}\,\psi^T\,\mathbf{A}\,\psi = \sum_{i=1}^{n} \left(k_i\,\frac{q_i^2}{2} + \frac{p_i^2}{2\,m_i} \right) .$$

Corresponding force matrix $\mathbf{F} = \gamma_0\,\mathbf{A}$:

$$\gamma_0 = \begin{pmatrix} 0 & \frac{1}{2\,m_1} & & & \\ -\frac{k_1}{2} & 0 & & & \\ & & 0 & \frac{1}{2\,m_2} & \\ & & -\frac{k_2}{2} & 0 & \dots \\ & & & \dots & \end{pmatrix}$$

With appropriate scaling of q_i and p_i: Decoupled force matrix **F** has the form:

$$\gamma_0 = \begin{pmatrix} 0 & \omega_1 & & & \\ -\omega_1 & 0 & & & \\ & & 0 & \omega_2 & \\ & & -\omega_2 & 0 & \dots \\ & & & \dots & \end{pmatrix}$$

We call this the standard form and the normalized standard form.

Symplectic Jacobi Transformations

Given a quadradic Hamiltonian in n dimensions:

- Strategy: Take 2 degrees of freedom (i.e. 4×4-submatrix) and ignore all other rows/columns.
- That is: We split the matrix in 2×2 blocks and diagonalize sequentially pairs of these blocks.
- As a consequence, it is sufficient to consider symplectic transformations of two degrees of freedom, i.e. the required matrices are 4×4.
- There are ten symplectic transformations in two dimensions. The structure of the 4×4 can be represented by the real Dirac matrices (RDMs) [3].
- We use the notation introduced as electromechanical equivalence (EMEQ) [3].

Why Dirac matrices? Aren't they only useful in Quantum mechanics?

- The system of the 16 real Dirac matrices is complete: Any *real*-valued 4×4-matrix $\mathbf{M}$ can be written as a linear combination of RDMs.
- The real matrix γ_0 can be identified with the symplectic unit matrix.
- All real Dirac matrices (RDMs) are either symplectic or "anti-symplectic".
- All RDMs square to $\pm\mathbf{1}$.
- All RDMs are either Hamiltonian or skew-Hamiltonian ("symplices" or "anti-symplices").
- All RDMs are either symmetric or skew-symmetric.
- All RDMs are either even (i.e. block-diagonal) or odd.

- All RDMs - except for the unit matrix - have zero trace.
- Two RDMs either commute or anti-commute.
- The RDMs form a group.
- The Hamiltonian RDMs ("symplices") are the generators of symplectic transformations. (A subset of these are the generators of Lorentz transformations.)

To conclude: The RDMs are **the** matrix-basis for coupling of symplectic systems.

Any arbitrary *real*-valued 4×4-matrix $\mathbf{M}$ as a linear combination of RDMs:

$$\mathbf{M} = \sum_{k=0}^{15} m_k \, \gamma_k \, .$$

The RDM-coefficients m_k can be computed by

$$m_k = sign(\gamma_k)\,Tr(\mathbf{M}\,\gamma_k + \gamma_k\,\mathbf{M})/8\,,$$

where $\mathrm{Tr}(X)$ is the trace of X and $sign(\gamma_k)$ is the *signature* of γ_k, i.e.:

$$sign(\gamma_k) = Tr(\gamma_k^2)/4 = \pm 1\,.$$

Note: Antisymmetric RDMs (γ_0,$\gamma_7 \dots \gamma_9$, γ_{10} and γ_{14}) have a negative signature, symmetric RDMs a positive.
Force matrices $\mathbf{F}$ must fulfill $\mathbf{F}^T = \gamma_0\,\mathbf{F}\,\gamma_0$ and are restricted to:

$$\mathbf{F} = \sum_{k=0}^{9} f_k\,\gamma_k\,.$$

The real Dirac matrices for the use in classical mechanics:

$$\gamma_\mu\,\gamma_\nu + \gamma_\nu\,\gamma_\mu = -2\,g_{\mu\nu} = 2\,\mathrm{Diag}(-1,1,1,1)\,.$$

(Dirac matrices with real numbers are only possible with the negative "metric tensor".)
The remaining matrices are defined by

$$\begin{array}{lclclcl}
\gamma_{14} & = & \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3; & & \gamma_{15} & = & \mathbf{1} \\
\gamma_4 & = & \gamma_0\,\gamma_1; & & \gamma_7 & = & \gamma_{14}\,\gamma_0\,\gamma_1 = \gamma_2\,\gamma_3 \\
\gamma_5 & = & \gamma_0\,\gamma_2; & & \gamma_8 & = & \gamma_{14}\,\gamma_0\,\gamma_2 = \gamma_3\,\gamma_1 \\
\gamma_6 & = & \gamma_0\,\gamma_3; & & \gamma_9 & = & \gamma_{14}\,\gamma_0\,\gamma_3 = \gamma_1\,\gamma_2 \\
\gamma_{10} & = & \gamma_{14}\,\gamma_0 & = & \gamma_1\,\gamma_2\,\gamma_3 & & \\
\gamma_{11} & = & \gamma_{14}\,\gamma_1 & = & \gamma_0\,\gamma_2\,\gamma_3 & & \\
\gamma_{12} & = & \gamma_{14}\,\gamma_2 & = & \gamma_0\,\gamma_3\,\gamma_1 & & \\
\gamma_{13} & = & \gamma_{14}\,\gamma_3 & = & \gamma_0\,\gamma_1\,\gamma_2 & &
\end{array}$$

Note: $\gamma_5 \neq \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3$ but instead $\gamma_{14} \equiv \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3$!

Define symplex to be the generator of a symplectic transformation. Basic symplices are $\gamma_0 \dots \gamma_9$.
The effect of a basic symplex γ_b is given by:

$$\begin{aligned}
\mathbf{R}_b &= \exp\left(\gamma_b\,\varepsilon/2\right) \\
&= \begin{cases} \mathbf{1}\,\cos\left(\varepsilon/2\right) + \gamma_b\,\sin\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = -1 \\ \mathbf{1}\,\cosh\left(\varepsilon/2\right) + \gamma_b\,\sinh\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = 1 \end{cases} \\
\mathbf{R}_b^{-1} &= \exp\left(-\gamma_b\,\varepsilon/2\right) \\
&= \begin{cases} \mathbf{1}\,\cos\left(\varepsilon/2\right) - \gamma_b\,\sin\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = -1 \\ \mathbf{1}\,\cosh\left(\varepsilon/2\right) - \gamma_b\,\sinh\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = 1 \end{cases}
\end{aligned}$$

Transformations with $\gamma_b^2 = -1$ are orthogonal transformations, i.e. *rotations* about angle ε, while those with $\gamma_b^2 = 1$ are *boosts* with "rapidity" ε.

Exactly analogue to transformation properties of Dirac spinors!

EMEQ: What is it? What is it good for?

- 10 components of symmetric 4×4-matrix $\mathbf{A}$ in Hamiltonian $H = \frac{1}{2}\,\psi^T\,\mathbf{A}\,\psi$.
- 10 components in force matrix $\mathbf{F} = \gamma_0\,\mathbf{A} = \sum\limits_{k=0}^{9} f_k\,\gamma_k$.
- 10 symplectic transformations generated by $\gamma_0 \dots \gamma_9$.
- 10 elements in σ-matrix.
- 10 physical quantities in relativistic electrodynamics: Energy + 3 x Momentum + 6 e.m. fields components.

EMEQ: Transformation properties of those 10 components are identical to those of then 10 physical quantities of Lorentz force equation: $\mathcal{E}$, $\vec{p}$, $\vec{E}$, $\vec{B}$ = 4 components of 4-vector + 6 “bivectors” of e.m. field. But: Let’s pretend that this equivalence is *formal* (for the moment). The components behave **as if**. Benefit: Meaningful nomenclature for the RDM-coefficients.

The most general force matrix **F** in two Dimensions can be written according to the EMEQ as:

$$\begin{aligned}\mathbf{F} &= \mathcal{E}\,\gamma_0 + \vec{P}\,\vec{\gamma} + \vec{E}\,\gamma_0\,\vec{\gamma} + \vec{B}\,\gamma_{14}\,\gamma_0\,\vec{\gamma} \\ &= \begin{pmatrix} -E_x & E_z + B_y & E_y - B_z & B_x \\ E_z - B_y & E_x & -B_x & -E_y - B_z \\ E_y + B_z & B_x & E_x & E_z - B_y \\ -B_x & -E_y + B_z & E_z + B_y & -E_x \end{pmatrix} \\ &+ \begin{pmatrix} -P_z & \mathcal{E} - P_x & 0 & p_y \\ -\mathcal{E} - P_x & P_z & -P_y & 0 \\ 0 & P_y & -P_z & \mathcal{E} + P_x \\ P_y & 0 & -\mathcal{E} + P_x & P_z \end{pmatrix}.\end{aligned}$$

Bringing **F** to block-diagonal form then means to find a symplectic transformation **R** with $\mathbf{F}' = \mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1}$ such that $\mathbf{F}'$ has standard form, or: $E'_x = E'_y = B'_x = B'_z = P'_y = P'_z = 0$.
Remaining: "Energy" $\mathcal{E}$, P_x, B_y and E_z. $\Rightarrow$ orthogonalization of $(\vec{P}, \vec{E}, \vec{B})$ required [5].

Symplectic Transformations II

$$\begin{aligned}
\mathcal{H}' &= \mathcal{H} \\
\psi' &= \mathbf{R}\,\psi \\
\bar{\psi}' &\equiv (\psi^T\,\gamma_0)' = \bar{\psi}\,\mathbf{R}^{-1} \\
\mathbf{F}' &= \mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1} \\
\mathbf{M}' &= \mathbf{R}\,\mathbf{M}\,\mathbf{R}^{-1}
\end{aligned}$$

The symplectic transformations generated by the ten RDMs $\gamma_0 \dots \gamma_9$ are

- γ_0: phase rotation.
- $\vec{\gamma}$ (γ_1, γ_2 and γ_3): phase boost along x, y, z.
- γ_4, γ_5 and γ_6: Lorentz boost along x, y, z.
- γ_7, γ_8 and γ_9: Spatial rotation about x-, y-, z-axis.

Rotations and Lorentz boosts are well known.

What do "phase rotations" and "phase boosts"?
The "phase rotation" $\mathbf{R}_0$ gives $\mathcal{E}' = \mathcal{E}$ and $\vec{B}' = \vec{B}$, but:

$$\begin{aligned} \vec{P}' &= \cos\varepsilon\,\vec{P} - \sin\varepsilon\,\vec{E} \\ \vec{E}' &= \cos\varepsilon\,\vec{E} + \sin\varepsilon\,\vec{P} \end{aligned}$$

The "phase boosts" $\mathbf{R}_k$ $k \in [1-3]$ are like Lorentz boosts, but with $\vec{E}$ and $\vec{P}$ exchanged. Hence they are identical to a sequence of 90° phase rotation + Lorentz boost + 90° inverse phase rotation.

- Lorentz boosts: $\vec{E}\,\vec{B}$ and $\vec{E}^2 - \vec{B}^2$ are invariant.
- Phase boosts: $\vec{P}\,\vec{B}$ and $\vec{P}^2 - \vec{B}^2$ are invariant.

The standard form of the decoupled force matrix $\mathbf{F}'$ is:

$$\begin{aligned}\mathbf{F}' &= \mathcal{E}'\,\gamma_0 + P_x'\,\gamma_1 + E_z'\,\gamma_6 + B_y'\,\gamma_8 \\ &= \begin{pmatrix} 0 & \mathcal{E}' - P_x' + E_z' + B_y' & 0 & 0 \\ -\mathcal{E}' - P_x' + E_z' - B_y' & 0 & 0 & 0 \\ 0 & 0 & 0 & E_z' - B_y' + \mathcal{E}' + P_x' \\ 0 & 0 & -\mathcal{E}' + P_x' + E_z' + B_y' & 0 \end{pmatrix}\end{aligned}$$

We use the abbreviations:

$$\begin{aligned} M_r &= \vec{E}\,\vec{B} \\ M_g &= \vec{B}\,\vec{P} \\ M_b &= \vec{E}\,\vec{P} \end{aligned}$$

and compute the transformation properties of these "mass components". The above scalar products are invariant under spatial rotations. Hence it is sufficient to consider boosts and the phase rotation.

$$\begin{aligned} \vec{r} &\equiv \mathcal{E}\,\vec{p} + \vec{B}\times\vec{E} \\ \vec{g} &\equiv \mathcal{E}\,\vec{E} + \vec{p}\times\vec{B} \\ \vec{b} &\equiv \mathcal{E}\,\vec{B} + \vec{E}\times\vec{p} \end{aligned}$$

	M_r'	M_g'	M_b'
γ_0	$M_r\,c + M_g\,s$	$M_g\,c - M_r\,s$	$M_b\,c_2 + \frac{\vec{P}^2-\vec{E}^2}{2}\,s_2$
γ_1	$M_r\,C - (\vec{b})_x\,S$	M_g	$M_b\,C - (\vec{r})_x\,S$
γ_2	$M_r\,C - (\vec{b})_y\,S$	M_g	$M_b\,C - (\vec{r})_y\,S$
γ_3	$M_r\,C - (\vec{b})_z\,S$	M_g	$M_b\,C - (\vec{r})_z\,S$
γ_4	M_r	$M_g\,C + (\vec{b})_x\,S$	$M_b\,C + (\vec{g})_x\,S$
γ_5	M_r	$M_g\,C + (\vec{b})_y\,S$	$M_b\,C + (\vec{g})_y\,S$
γ_6	M_r	$M_g\,C + (\vec{b})_z\,S$	$M_b\,C + (\vec{g})_z\,S$

$$\begin{aligned} c &= \cos(\varepsilon) & s &= \sin(\varepsilon) \\ c_2 &= \cos(2\,\varepsilon) & s_2 &= \sin(2\,\varepsilon) \\ C &= \cosh(\varepsilon) & S &= \sinh(\varepsilon) \\ C_2 &= \cosh(2\,\varepsilon) & S_2 &= \sinh(2\,\varepsilon) \end{aligned}$$

The Decoupling Algorithm

$$\begin{aligned} \mathbf{F} &\rightarrow \mathbf{R}_b\, \mathbf{F}\, \mathbf{R}_b^{-1} \\ \mathbf{R}_b &= \exp\left(\gamma_b\, \varepsilon/2\right) \end{aligned}$$

- $M_g \rightarrow 0$: Make phase rotation using γ_0 and angle $\varepsilon = \arctan\left(\frac{M_g}{M_r}\right)$.
- $\vec{b} \rightarrow |\vec{b}|\,\vec{e}_y$: Align the vector $\vec{b}$ along the y-axis by the spatial rotations with γ_7 with $\varepsilon = \arctan\left(\frac{b_z}{b_y}\right)$ and with γ_9 with $\varepsilon = -\arctan\left(\frac{b_x}{b_y}\right)$. Such rotations can always be done.
- $M_r \rightarrow 0$: Boost using γ_2 and rapidity $\varepsilon = \mathrm{artanh}(\frac{M_r}{b_y})$.

The last transformation is only possible, if $|M_r| < |b_y| = |\vec{b}|$:

$$(\vec{E}\,\vec{B})^2 \leq -2\,\mathcal{E}\,\vec{p}\,(\vec{E}\times\vec{B}) + \mathcal{E}^2\,\vec{B}^2 + \vec{E}^2\,\vec{P}^2 - (\vec{E}\,\vec{P})^2$$

After the first transformation we have $M_g = \vec{P}\,\vec{B} = 0$.

The eigenvalues of the force matrix **F** are

$$
\begin{aligned}
\lambda &= \mathrm{Diag}(i\,\omega_1, -i\,\omega_1, i\,\omega_2, -i\,\omega_2) \\
K_1 &= \mathcal{E}^2 + \vec{B}^2 - \vec{E}^2 - \vec{P}^2 \\
K_2 &= -2\,\mathcal{E}\,\vec{P}\,(\vec{E}\times\vec{B}) + \mathcal{E}^2\,\vec{B}^2 + \vec{E}^2\,\vec{P}^2 \\
&- (\vec{E}\,\vec{P})^2 - (\vec{E}\,\vec{B})^2 - (\vec{P}\,\vec{B})^2 \\
\omega_1 &= \sqrt{K_1 + 2\,\sqrt{K_2}} \\
\omega_2 &= \sqrt{K_1 - 2\,\sqrt{K_2}} \\
\mathrm{Det}(\mathbf{F}) &= K_1^2 - 4\,K_2
\end{aligned}
$$

System stable $\Rightarrow$ eigenfrequencies are real:

$$
\begin{aligned}
K_2 &\geq 0 \\
K_1 &\geq 2\,\sqrt{K_2}
\end{aligned}
$$

The requirement $|M_r| < |b_y|$ is equal to $K_2 \geq 0$. Eigenvalues must be real or imaginary, but not off axis in the complex plane. Then the vector-components $(\vec{g})_y$ and $(\vec{r})_y$, $(\vec{b})_x$ and $(\vec{b})_z$ are zero after the decoupling transformations have been applied. With $M_r = M_g = 0$ we can align $\vec{B}$ along y-axis so that $\vec{B} = |\vec{B}|\,\vec{e}_y$ and obtain in total $B_x = B_z = E_y = P_y = 0$. The transformed force matrix is then block-diagonal:

$$\mathbf{F}' = \begin{pmatrix} -E_x - P_z & \mathcal{E} - P_x + E_z + B_y & 0 & 0 \\ -\mathcal{E} - P_x + E_z - B_y & E_x + P_z & 0 & 0 \\ 0 & 0 & E_x - P_z & E_z - B_y + \mathcal{E} + P_x \\ 0 & 0 & -\mathcal{E} + P_x + E_z + B_y & P_z - E_x \end{pmatrix}$$

The Decoupling Algorithm III

In order to bring the block-diagonal force matrix to standard form, apply the following transformations:

- $M_b \to 0$: Use another phase rotation with γ_0 about $\varepsilon = \frac{1}{2}\,\arctan\left(\frac{2\,M_b}{\vec{E}^2 - \vec{P}^2}\right)$
- $P_z \to 0$: Use rotation about y-axis with γ_8 about $\varepsilon = -\arctan\left(\frac{P_z}{P_x}\right)$.

After these two rotations, the matrix has normal form, if $K_2 > 0$ holds. In ion beam optics this is usually the case and therefore we consider this method as a generally applicable decoupling algorithm.

Decoupling the Transfer Matrix I

- Up to now we always referred to the (average) force matrix **F**.
- What if we don't have the force, but only the (one-turn-) transfer matrix **M**?

We have:

$$
\begin{aligned}
\mathbf{F} &= \mathbf{E}\,\lambda\,\mathbf{E}^{-1} \\
\lambda &= \mathrm{Diag}(i\,\omega_1, -i\,\omega_1, i\,\omega_2, -i\,\omega_2) \\
&= -i\,\frac{\omega_1+\omega_2}{2}\,\gamma_3 - i\,\frac{\omega_1-\omega_2}{2}\,\gamma_4 \\
\mathbf{M} &= \mathbf{E}\,\Lambda\,\mathbf{E}^{-1} \\
\Lambda &= \exp\left(\lambda\,\tau\right) \\
&= \mathrm{Diag}(e^{i\,\omega_1\,\tau}, e^{-i\,\omega_1\,\tau}, e^{i\,\omega_2\,\tau}, e^{-i\,\omega_2\,\tau})
\end{aligned}
$$

Define

$$\bar{\omega} = \frac{\omega_1+\omega_2}{2} \qquad \Delta\omega = \frac{\omega_1-\omega_2}{2}$$

so that RDM-coefficients of Λ:

$$\begin{aligned}
\Sigma_c &= \frac{\cos(\omega_1\,\tau)+\cos(\omega_2\,\tau)}{2} = \cos(\bar{\omega}\,\tau)\,\cos(\Delta\omega\,\tau)\\
\Sigma_s &= \frac{\sin(\omega_1\,\tau)+\sin(\omega_2\,\tau)}{2} = \sin(\bar{\omega}\,\tau)\,\cos(\Delta\omega\,\tau)\\
\Delta_s &= \frac{\sin(\omega_1\,\tau)-\sin(\omega_2\,\tau)}{2} = \cos(\bar{\omega}\,\tau)\,\sin(\Delta\omega\,\tau)\\
\Delta_c &= \frac{\cos(\omega_1\,\tau)-\cos(\omega_2\,\tau)}{2} = -\sin(\bar{\omega}\,\tau)\,\sin(\Delta\omega\,\tau)
\end{aligned}$$

and then:

$$\Lambda = \Sigma_c\,\mathbf{1} - i\,\Sigma_s\,\gamma_3 - i\,\Delta_s\,\gamma_4 - \Delta_c\,\gamma_{12}\,,$$

Then the transfer matrix can be written as:

$$\begin{aligned} \mathbf{M} &= \Sigma_c\,\mathbf{1} - i\,\Sigma_s\,\mathbf{E}\,\gamma_3\,\mathbf{E}^{-1} - i\,\Delta_s\,\mathbf{E}\,\gamma_4\,\mathbf{E}^{-1} - \Delta_c\,\mathbf{E}\,\gamma_{12}\,\mathbf{E}^{-1} \\ \mathbf{F} &= -i\,\bar{\omega}\,\mathbf{E}\,\gamma_3\,\mathbf{E}^{-1} - i\,\Delta\omega\,\mathbf{E}\,\gamma_4\,\mathbf{E}^{-1} \end{aligned}$$

- Comparison results: A “part” of the transfer matrix is structurally identical to force matrix, but has different eigenvalues.
- Decoupling method does not directly depend on the eigenvalues.
- $\Rightarrow$ this part of the transfer matrix can be decoupled with same method as force matrix.
- Surprise, surprise: The remaining “part” will be decoupled with the same transformations.

Split transfer matrix:

$$\begin{aligned}\frac{1}{2}\,(\mathbf{M}\pm\mathbf{M}^{-1}) &= \mathbf{E}\,\frac{\Lambda(\tau)\pm\Lambda(-\tau)}{2}\,\mathbf{E}^{-1}\\ &= \frac{1}{2}\,(\mathbf{M}\mp\gamma_0\,\mathbf{M}^T\,\gamma_0)\end{aligned}$$

so that:

$$\begin{aligned}\mathbf{M}_s &= \frac{1}{2}\,(\mathbf{M}+\gamma_0\,\mathbf{M}^T\,\gamma_0)=\sum_{k=0}^{9}\,m_k\,\gamma_k\\ &= -i\,\Sigma_s\,\mathbf{E}\,\gamma_3\,\mathbf{E}^{-1}-i\,\Delta_s\,\mathbf{E}\,\gamma_4\,\mathbf{E}^{-1}\\ \mathbf{M}_c &= \frac{1}{2}\,(\mathbf{M}-\gamma_0\,\mathbf{M}^T\,\gamma_0)=\sum_{k=10}^{15}\,m_k\,\gamma_k\\ &= \Sigma_c\,\mathbf{1}-\Delta_c\,\mathbf{E}\,\gamma_{12}\,\mathbf{E}^{-1}\end{aligned}$$

Conclusion: We split the transfer matrix and use only the first 10 RDMs for decoupling. The rest can be (and has to be) ignored.

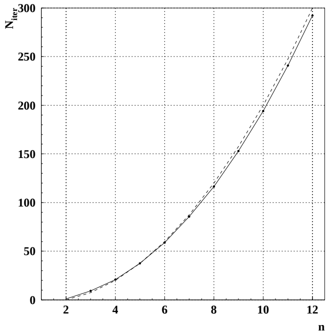

Figure: Solid line: Number of iterations required to bring a $2\,n \times 2\,n$ symplex (Hamiltonian matrix) to normal form. Dashed line: Approximation by $5\,\frac{n\,(n-2)}{2}$. The number n_b of non-diagonal 2×2-blocks is $n_b = \frac{n\,(n-1)}{2}$.

Fig. 1 shows the average number of iterations that is required to compute the transformation that brings a $2\,n \times 2\,n$ symplex to standard form.

Performance II

$$\begin{pmatrix} 0.0487 & 3.4674 & -0.2785 & 0.2587 & 0.1029 & -0.2109 \\ -3.4390 & -0.0487 & 0.4615 & -0.3062 & -0.3376 & 0.3446 \\ 0.3062 & 0.2587 & 0.3280 & 3.4698 & 0.3186 & -0.2946 \\ 0.4615 & 0.2785 & -3.5757 & -0.3280 & 0.1402 & 0.3129 \\ -0.3446 & -0.2109 & -0.3129 & -0.2946 & -0.2855 & 3.5500 \\ -0.3376 & -0.1029 & 0.1402 & -0.3186 & -3.8665 & 0.2855 \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} -0.1484 & 3.9725 & -0.0000 & 0.0000 & 0.4744 & -0.2566 \\ -3.6134 & 0.1484 & -0.0000 & 0.0000 & -0.1289 & 0.2831 \\ 0.0000 & -0.0000 & -0.0345 & 3.6508 & 0.0358 & -0.3001 \\ -0.0000 & 0.0000 & -2.6903 & 0.0345 & -0.0228 & 0.3030 \\ -0.2831 & -0.2566 & -0.3030 & -0.3001 & -0.2855 & 3.5500 \\ -0.1289 & -0.4744 & -0.0228 & -0.0358 & -3.8665 & 0.2855 \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} -0.2862 & 4.3793 & 0.1553 & 0.1594 & -0.0000 & -0.0000 \\ -3.8828 & 0.2862 & -0.1261 & -0.1181 & 0.0000 & 0.0000 \\ 0.1181 & 0.1594 & -0.0345 & 3.6508 & 0.1788 & -0.1359 \\ -0.1261 & -0.1553 & -2.6903 & 0.0345 & -0.1739 & 0.1448 \\ -0.0000 & -0.0000 & -0.1448 & -0.1359 & 0.1125 & 3.4495 \\ 0.0000 & 0.0000 & -0.1739 & -0.1788 & -3.2672 & -0.1125 \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} -0.2862 & 4.3793 & 0.0891 & -0.0493 & 0.1045 & 0.1633 \\ -3.8828 & 0.2862 & -0.0677 & 0.0419 & -0.0871 & -0.1230 \\ -0.0419 & -0.0493 & 0.0285 & 3.3826 & -0.0000 & 0.0000 \\ -0.0677 & -0.0891 & -3.5022 & -0.0285 & 0.0000 & -0.0000 \\ 0.1230 & 0.1633 & 0.0000 & 0.0000 & -0.3288 & 3.3253 \\ -0.0871 & -0.1045 & 0.0000 & 0.0000 & -2.8103 & 0.3288 \end{pmatrix} \tag{4}$$

Performance II

$$\begin{pmatrix} -0.1458 & 4.4771 & 0.0745 & -0.0404 & 0.0000 & 0.0000 \\ -3.8103 & 0.1458 & -0.0827 & 0.0501 & 0.0000 & -0.0000 \\ -0.0501 & -0.0404 & 0.0285 & 3.3826 & 0.0064 & 0.0062 \\ -0.0827 & -0.0745 & -3.5022 & -0.0285 & 0.0107 & 0.0111 \\ 0.0000 & 0.0000 & -0.0111 & 0.0062 & -0.2413 & 3.4096 \\ -0.0000 & -0.0000 & 0.0107 & -0.0064 & -2.6930 & 0.2413 \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} -0.3160 & 3.9883 & -0.0000 & 0.0000 & -0.0011 & -0.0011 \\ -4.3094 & 0.3160 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.0511 & 3.4809 & -0.0034 & -0.0039 \\ -0.0000 & -0.0000 & -3.3898 & -0.0511 & 0.0119 & 0.0120 \\ 0.0000 & -0.0011 & -0.0120 & -0.0039 & -0.2413 & 3.4096 \\ 0.0000 & 0.0011 & 0.0119 & 0.0034 & -2.6930 & 0.2413 \end{pmatrix} \tag{6}$$

Performance II

$$\begin{pmatrix} -0.3160 & 3.9883 & 0.0000 & 0.0000 & 0.0007 & -0.0014 \\ -4.3094 & 0.3160 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0683 & 3.4297 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -3.4414 & 0.0683 & 0.0000 & -0.0000 \\ -0.0000 & -0.0014 & 0.0000 & 0.0000 & -0.0192 & 2.6195 \\ 0.0000 & -0.0007 & -0.0000 & 0.0000 & -3.4827 & 0.0192 \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} -0.0152 & 3.7947 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -4.5030 & 0.0152 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0683 & 3.4297 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -3.4414 & 0.0683 & 0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3666 & 2.8225 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -3.2797 & -0.3666 \end{pmatrix} \tag{8}$$

Performance II

$$\begin{pmatrix} -0.3432 & 4.2375 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -4.0602 & 0.3432 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0085 & 3.5035 & -0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -3.3675 & -0.0085 & 0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3666 & 2.8225 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -3.2797 & -0.3666 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} -0.3432 & 4.2375 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -4.0602 & 0.3432 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0161 & 3.3689 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -3.5022 & -0.0161 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.4237 & 3.1353 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & -2.9669 & 0.4237 \end{pmatrix} \tag{10}$$

Performance II

$$\begin{pmatrix} -0.1563 & 4.4670 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -3.8307 & 0.1563 & 0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0161 & 3.3689 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & -3.5022 & -0.0161 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.2121 & 3.4275 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & -2.6747 & 0.2121 \end{pmatrix} \tag{11}$$

Summary

- The general structure of symplectic coupling has been described using the formalism of the real Dirac matrices (RDMs).
- A straightforward method of decoupling has been derived.
- The electromechanical equivalence (EMEQ) allows to use a familiar and meaningful nomenclature.
- Regular and irregular systems have been described.
- Space charge dominated coupling in isochronous cyclotrons is an example for an irregular system.

Referenzen

[1] V.I. Arnold: Mathematical Methods of Classical Mechanics; 2nd Ed., Springer, New York 2010.

[2] Andrej Wolski: Alternative Approach to General Coupled Linear Optics; PRST-AB 9, 024001 (2006).

[3] C. Baumgarten; Phys. Rev. ST Accel. Beams. 14, 114002 (2011); http://arxiv.org/abs/1109.2001.

[4] C. Baumgarten; Phys. Rev. ST Accel. Beams. 14, 114201 (2011); http://arxiv.org/abs/1109.1996.

[5] C. Baumgarten; submitted to Phys. Rev. ST Accel. Beams; http://arxiv.org/abs/1201.0907.