Algebraic CST_2: Real Dirac Matrices

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Hamilton function with symmetric matrix $\mathbf{A}$:

$$\mathcal{H} = \frac{1}{2}\,\psi^T\,\mathbf{A}\,\psi\,,$$

where $\psi = (q_1, p_1, \dots, q_n, p_n)^T$. Equations of Motion (EQOM):

$$\begin{aligned} \dot{q}_i &= \frac{\partial\mathcal{H}}{\partial p_i} & \dot{p}_i &= -\frac{\partial\mathcal{H}}{\partial q_i} \\ \dot{\psi} &= \gamma_0\,\nabla_\psi\,\mathcal{H} \\ &= \gamma_0\,\mathbf{A}\,\psi = \mathbf{F}\,\psi\,, \end{aligned}$$

where γ_0 has a blockdiagonal skew-symmetric structure:

$$\gamma_0 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \dots \\ & & & \dots & \end{pmatrix}$$

Hamilton EQOM:

$$\begin{aligned} \dot{\psi} &= \gamma_0\,\mathbf{A}\,\psi = \mathbf{F}\,\psi \\ \mathbf{F}^T &= (\gamma_0\,\mathbf{A})^T = -\mathbf{A}\,\gamma_0 = \gamma_0\,\mathbf{F}\,\gamma_0 \end{aligned}$$

since $\gamma_0^T = -\gamma_0$ and $\gamma_0^2 = -1$.

We call a matrix **F** that holds

$$\mathbf{F}^T = \gamma_0\,\mathbf{F}\,\gamma_0$$

a **symplex**. Also called "infinitesimal symplectic" or "Hamiltonian matrix".

Matrix of second moments σ by RMS:

$$\begin{aligned} \sigma &\equiv \frac{1}{N}\sum_{i=1}^{N}\psi_i\,\psi_i^T \\ \mathbf{S} &\equiv \sigma\,\gamma_0 \end{aligned}$$

EQOM of second moments:

$$\begin{aligned} \dot\sigma &= \frac{1}{N}\sum_{i=1}^{N}(\dot\psi_i\,\psi_i^T+\psi_i\,\dot\psi_i^T) \\ &= \mathbf{F}\,\sigma+\sigma\,\mathbf{F}^T \\ &= \mathbf{F}\,\sigma+\sigma\,\gamma_0\,\mathbf{F}\,\gamma_0 \end{aligned}$$

Multiplication with γ_0 from the right side:

$$\begin{aligned} \dot\sigma\,\gamma_0 &= \mathbf{F}\,\sigma\,\gamma_0-\sigma\,\gamma_0\,\mathbf{F} \\ \dot{\mathbf{S}} &= \mathbf{F}\,\mathbf{S}-\mathbf{S}\,\mathbf{F} \end{aligned}$$

Solution of EQOM for constant **F**

If the matrix **F** is constant within $[0 \dots \tau]$, then:

$$\begin{aligned}
\dot{\psi} &= \mathbf{F}\,\psi \\
\psi(\tau) &= \exp\left(\int\limits_0^\tau \mathbf{F}\,dt\right)\psi(0) \\
&= \exp\left(\mathbf{F}\,\tau\right)\psi(0) \\
&= \mathbf{M}(\tau,0)\,\psi(0)
\end{aligned}$$

Hence the solution of a piecewise constant **F** can be written as a product of the corresponding transfer matrices. The σ-matrix transforms accordingly:

$$\begin{aligned}
\sigma(\tau) &= \frac{1}{N}\sum_{i=0}^{N}\psi_i(\tau)\,\psi_i^T(\tau) \\
&= \frac{1}{N}\sum_{i=0}^{N}\mathbf{M}\,\psi_i(0)\,\psi_i^T(0)\,\mathbf{M}^T \\
&= \mathbf{M}\,\sigma(0)\,\mathbf{M}^T
\end{aligned}$$

Example: $(q1, p1, q2, p2) \equiv (x, x', y, y')$, i.e. transversal coordinates in a horizontally focusing quadrupole:

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & k^2 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{M} &= \exp\left(\mathbf{F}\,\tau\right) \\ &= \begin{pmatrix} \cos k\,\tau & \frac{\sin k\,\tau}{k} & 0 & 0 \\ -k\,\sin k\,\tau & \cos k\,\tau & 0 & 0 \\ 0 & 0 & \cosh k\,\tau & \frac{\sinh k\,\tau}{k} \\ 0 & 0 & k\,\sinh k\,\tau & \cosh k\,\tau \end{pmatrix} \end{aligned}$$

$$\begin{array}{rr} & \mathbf{F}^T = \gamma_0\,\mathbf{F}\,\gamma_0 \\ \Rightarrow & \mathbf{M}\,\gamma_0\,\mathbf{M}^T = \gamma_0 \end{array}$$

Proof:

$$\begin{aligned}
\mathbf{M}\,\gamma_0\,\mathbf{M}^T &= \exp(\mathbf{F}\,\tau)\,\gamma_0\,(\exp(\mathbf{F}\,\tau))^T \\
&= \exp(\mathbf{F}\,\tau)\,\gamma_0\,\left(\sum_{k=0}^{\infty}\frac{(\mathbf{F}^T)^k\,\tau^k}{k!}\right) \\
&= \exp(\mathbf{F}\,\tau)\,\gamma_0\,\left(\sum_{k=0}^{\infty}\frac{(\gamma_0\,\mathbf{F}\,\gamma_0)^k\,\tau^k}{k!}\right) \\
&= \exp(\mathbf{F}\,\tau)\,\gamma_0\,\left(1+\frac{\gamma_0\,\mathbf{F}\,\gamma_0}{1}+\frac{\gamma_0\,\mathbf{F}\,\gamma_0\,\gamma_0\,\mathbf{F}\,\gamma_0}{2!}+\dots\right) \\
&= \exp(\mathbf{F}\,\tau)\,\left(1-\frac{\mathbf{F}}{1}+\frac{\mathbf{F}^2}{2!}-\frac{\mathbf{F}^3}{3!}+\dots\right)\,\gamma_0 \\
&= \exp(\mathbf{F}\,\tau)\,\exp(-\mathbf{F}\,\tau)\,\gamma_0 \\
&= \gamma_0
\end{aligned}$$

Eigenvalues I

The Courant Snyder theory is about **stable** systems. $\Rightarrow$ The eigenvalues of transfer matrices representing "strongly" stable systems are complex conjugate pairs on the unit circle in the complex plane [1]:

$$\begin{aligned} \mathbf{M}(s) &= \mathbf{E}\,\Lambda(s)\,\mathbf{E}^{-1} \\ \Lambda &= \mathrm{Diag}(e^{i\,\omega_1\,s}, e^{-i\,\omega_1\,s}, e^{i\,\omega_2\,s}, e^{-i\,\omega_2\,s}) \end{aligned}$$

This means that strongly stable systems can be represented by a constant force matrix $\bar{\mathbf{F}}$ with purely imaginary eigenvalues:

$$\begin{aligned} \bar{\mathbf{F}} &= \mathbf{E}\,\lambda\,\mathbf{E}^{-1} \\ \lambda &= \mathrm{Diag}(i\,\omega_1, -i\,\omega_1, i\,\omega_2, -i\,\omega_2) \end{aligned}$$

where

$$\Lambda = \exp\left(\lambda\, s\right)$$

(Easy proof.)

A system with a one-turn (one-sector) transfer matrix **M** is matched, if

$$
\begin{aligned}
\sigma(s) &= \mathbf{M}\,\sigma(0)\,\mathbf{M}^T \\
\sigma &= -\mathbf{M}\,\sigma\,\gamma_0\,\mathbf{M}^{-1}\,\gamma_0 \\
\sigma\,\gamma_0 &= \mathbf{M}\,\sigma\,\gamma_0\,\mathbf{M}^{-1} \\
\mathbf{S} &= \mathbf{M}\,\mathbf{S}\,\mathbf{M}^{-1} \\
\mathbf{S}\,\mathbf{M} &= \mathbf{M}\,\mathbf{S} \\
\mathbf{S}\,\mathbf{E}\,\Lambda\,\mathbf{E}^{-1} &= \mathbf{E}\,\Lambda\,\mathbf{E}^{-1}\,\mathbf{S} \\
(\mathbf{E}^{-1}\,\mathbf{S}\,\mathbf{E})\,\Lambda &= \Lambda\,(\mathbf{E}^{-1}\,\mathbf{S}\,\mathbf{E}) \\
\mathbf{D}\,\Lambda &= \Lambda\,\mathbf{D}
\end{aligned}
$$

Eigenvalues III (Matching)

Since only a diagonal matrix commutes with diagonal matrices, **D** is diagonal, i.e.:

$$\begin{aligned} \mathbf{D} &= \mathbf{E}^{-1}\,\mathbf{S}\,\mathbf{E} \\ \mathbf{S} &= \mathbf{E}\,\mathbf{D}\,\mathbf{E}^{-1} \end{aligned}$$

A. Wolski [2]:

$$\mathbf{D} = \mathrm{Diag}(i\,\varepsilon_1, -i\,\varepsilon_1, i\,\varepsilon_2, -i\,\varepsilon_2)\,.$$

$\Rightarrow$ Find matched distribution σ_m for given emittances ε_i by the determination of the matrix of eigenvectors **E**:

$$\sigma_m = -\mathbf{E}\,\mathbf{D}\,\mathbf{E}^{-1}\,\gamma_0\,.$$

Or (equivalent) find transformation such that $\bar{\mathbf{F}}$ and hence **M** are block-diagonal. Then it is possible to apply 1-dim. Courant-Snyder-Theory (CST).

Why Real Dirac Matrices?

Why introducing Real Dirac Matrices in Classical Mechanics?

1. The RDMs form a complete system of ("normalized") matrices. This is useful in the same way as orthogonal coordinate systems are.
2. The RDMs are easy to handle, have zero trace (only the unit matrix has a non-zero trace) and form a group.
3. The RDMs have unique features: All RDMs are either symplectic of antisymplectic, a symplex or an antisymplex, either odd or even.
4. Real because the 16 real matrices form a complete system.
5. Real because matrices **F**, **M** and **S** and all variables are real.

The *real* Dirac matrices do only exist for "signature" $(-,+,+,+)$. We define γ_0 to be the symplectic unit matrix, i.e. it is defined by the Hamilton EQOM:

$$\dot{\psi} = \gamma_0\,\mathbf{A}\,\psi\,.$$

Then 3 more basic matrices γ_k $k \in [1,2,3]$ are to be found with

$$\gamma_0\,\gamma_k + \gamma_k\,\gamma_0 = 0$$

A possible choice for $\psi = (q_1, p_1, q_2, p_2)^T$ is:

$$\begin{aligned}
\gamma_0 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} &
\gamma_1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
\gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} &
\gamma_3 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

The remaining matrices are defined by

$$
\begin{array}{rclcrcl}
\gamma_{14} &=& \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3; & & \gamma_{15} &=& \mathbf{1} \\
\gamma_4 &=& \gamma_0\,\gamma_1; & & \gamma_7 &=& \gamma_{14}\,\gamma_0\,\gamma_1 = \gamma_2\,\gamma_3 \\
\gamma_5 &=& \gamma_0\,\gamma_2; & & \gamma_8 &=& \gamma_{14}\,\gamma_0\,\gamma_2 = \gamma_3\,\gamma_1 \\
\gamma_6 &=& \gamma_0\,\gamma_3; & & \gamma_9 &=& \gamma_{14}\,\gamma_0\,\gamma_3 = \gamma_1\,\gamma_2 \\
\gamma_{10} &=& \gamma_{14}\,\gamma_0 &=& \gamma_1\,\gamma_2\,\gamma_3 & & \\
\gamma_{11} &=& \gamma_{14}\,\gamma_1 &=& \gamma_0\,\gamma_2\,\gamma_3 & & \\
\gamma_{12} &=& \gamma_{14}\,\gamma_2 &=& \gamma_0\,\gamma_3\,\gamma_1 & & \\
\gamma_{13} &=& \gamma_{14}\,\gamma_3 &=& \gamma_0\,\gamma_1\,\gamma_2 & &
\end{array}
$$

Note: $\gamma_5 \neq \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3$ but instead $\gamma_{14} \equiv \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3$!

The RDMs (as used in Dirac equation) have an intrinsic structure:

- The four basic matrices $\gamma_0 \dots \gamma_3$ represent a 4-vector.
- The six products of two basic matrices are called "bi-vectors" $\gamma_4 \dots \gamma_9$.
- The four products of 3 basic matrices are a "pseudo-4-vector" $\gamma_{10} \dots \gamma_{13}$.
- The product of all four basic matrices is the pseudo-scalar: $\gamma_{14} = \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3$.
- The unit matrix γ_{15} is a scalar.

This structure refers to the *transformation properties* in Minkowski space and allows a certain grouping of the RDMs (see for instance Messiah, Quantum Mechanics II).

γ_x	E.M.	γ_x^2	γ_x^T	s/a	S/A	$\tilde{\gamma}_0$	e/o
γ_0	E	-1	$-$	s	S	y	e
$\vec{\gamma}$	$\vec{P}$	$+1$	$+$	s	A	n	(e,o,e)
$\gamma_0\vec{\gamma}$	$\vec{E}$	$+1$	$+$	s	A	n	(e,o,e)
$\gamma_{14}\gamma_0\vec{\gamma}$	$\vec{B}$	-1	$-$	s	S	y	(o,e,o)
$\gamma_{14}\gamma_0$		-1	$-$	a	A	y	o
$\gamma_{14}\vec{\gamma}$		$+1$	$+$	a	S	n	(o,e,o)
γ_{14}		-1	$-$	a	A	y	o
$\gamma_{15} = \mathbf{1}$		$+1$	$+$	a	S	n	e

"s/a" **s**ymplex or **a**ntisymplex.

"S/A" **s**ymplectic or **a**ntisymplectic.

"e/o" **e**ven or **o**dd.

$\tilde{\gamma}_0$ A basis exists (y) or not (n) in which γ_x is the "time direction".

Any *real* 4×4-matrix $\mathbf{M}$ can be written as a linear combination of RDMs:

$$\mathbf{M} = \sum_{k=0}^{15} m_k \, \gamma_k \, .$$

The RDM-coefficients m_k can be computed by

$$m_k = sign(\gamma_k) \, Tr(\mathbf{M} \, \gamma_k + \gamma_k \, \mathbf{M})/8 \, ,$$

where $sign(\gamma_k)$ is the *signature* of γ_k, i.e.:

$$sign(\gamma_k) = Tr(\gamma_k^2)/4 = \pm 1 \, .$$

Note: Antisymmetric RDMs (γ_0,$\gamma_7 \dots \gamma_9$, γ_{10} and γ_{14}) have a negative signature, symmetric RDMs a positive.

- A **symplex F** in CST_n fulfills $\mathbf{F}^T = \gamma_0\,\mathbf{F}\,\gamma_0$.
- The product of two anticommuting symplices is a symplex.
- CST_2: The basic matrices γ_0, γ_1, γ_2 and γ_3 are anticommuting symplices.

$\Rightarrow$ All products of two basic matrices are symplices: $\gamma_4 \dots \gamma_9$.

- Six such products exists $(\gamma_4 \dots \gamma_9)$ plus four basic matrices.

$\Rightarrow$ $\nu = 10$ parameters of the 2-dim. force matrix are found.

$\Rightarrow$ The force matrix **F** can then be written as:

$$\mathbf{F} = \sum_{k=0}^{9} f_k\,\gamma_k\,.$$

A linear coordinate transformation represented by a matrix $\mathbf{R}$ is called symplectic, if $\mathbf{R}\,\gamma_0\,\mathbf{R}^T = \gamma_0$ holds. In this case, the Hamiltonian is invariant and:

$$
\begin{aligned}
\mathcal{H}' &= \mathcal{H} \\
\psi' &= \mathbf{R}\,\psi \\
(\psi^T)' &= \psi^T\,\mathbf{R}^T = -\psi^T\,\gamma_0\,\mathbf{R}^{-1}\,\gamma_0 \\
(\psi^T\,\gamma_0)' &= \psi^T\,\gamma_0\,\mathbf{R}^{-1} \\
\mathbf{F}' &= \mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1} \\
\mathbf{M}' &= \mathbf{R}\,\mathbf{M}\,\mathbf{R}^{-1} \\
\mathbf{S}' &= \mathbf{R}\,\mathbf{S}\,\mathbf{R}^{-1}
\end{aligned}
$$

The RDM's **and the Hamilton equations of motion** have the same form in all coordinate systems. The transfer matrix $\mathbf{M}$ is a symplectic transformation matrix. (i.e. the Hamiltonian matrix $\mathbf{F}$ is the generator of a symplectic transformation).

Symplectic Transformations I

Symplectic transformations are given by (products of) matrix-exponentials of symplices. There are ten **basic** symplices, i.e. γ_k with $k \in [0-9]$:

$$\mathbf{F} = \sum_{k=0}^{9} f_k\,\gamma_k\,.$$

Hence we have ten possible symplectic transformations controlled by a (dimensionless) parameter ε:

$$\begin{aligned} \mathbf{R}_k(\varepsilon) &= \exp\left(\gamma_k\,\varepsilon/2\right) \\ \mathbf{R}_k^{-1}(\varepsilon) &= \exp\left(-\gamma_k\,\varepsilon/2\right) = \mathbf{R}_k(-\varepsilon) \end{aligned}$$

Consider we transform a matrix $\mathbf{X} = \mathcal{E}\,\gamma_0 + P_x\,\gamma_1 + P_y\,\gamma_2 + P_z\,\gamma_3$ using $\mathbf{R}_4$, then we obtain after decomposition into the RDM-coefficients:

$$\begin{aligned}
\mathbf{X}' &= \mathbf{R}\,\mathbf{X}\,\mathbf{R}^{-1}\\
\mathbf{X}' &= \mathcal{E}'\,\gamma_0 + P_x'\,\gamma_1 + P_y'\,\gamma_2 + P_z'\,\gamma_3\\
\mathcal{E}' &= \mathcal{E}\,\cosh(\varepsilon) + P_x\,\sinh(\varepsilon)\\
P_x' &= P_x\,\cosh(\varepsilon) + \mathcal{E}\,\sinh(\varepsilon)\\
P_y' &= P_y\\
P_z' &= P_z
\end{aligned}$$

Using the usual parametrization in Minkowski space where ε is the "rapidity" ($\beta = \tanh(\varepsilon)$, $\gamma = \cosh(\varepsilon)$, $\beta\gamma = \sinh(\varepsilon)$), we find

$$\begin{aligned}
\mathcal{E}' &= \gamma\,\mathcal{E} + \beta\,\gamma\,P_x\\
P_x' &= \gamma\,P_x + \beta\,\gamma\,\mathcal{E}
\end{aligned}$$

which is evidently identical to a Lorentz boost along x.

Same transformation, but
$\mathbf{X} = E_x\,\gamma_4 + E_y\,\gamma_5 + E_z\,\gamma_6 + B_x\,\gamma_7 + B_y\,\gamma_8 + B_z\,\gamma_9$:

$$\begin{aligned}
\mathbf{X}' &= \mathbf{R}\,\mathbf{X}\,\mathbf{R}^{-1} \\
E_x' &= E_x \\
B_x' &= B_x \\
E_y' &= E_y\,\cosh(\varepsilon) + B_z\,\sinh(\varepsilon) \\
E_z' &= E_z\,\cosh(\varepsilon) - B_y\,\sinh(\varepsilon) \\
B_y' &= B_y\,\cosh(\varepsilon) - E_z\,\sinh(\varepsilon) \\
B_z' &= B_z\,\cosh(\varepsilon) + E_y\,\sinh(\varepsilon)
\end{aligned}$$

which is again identical to a Lorentz boost along x, but now of the e.m. fields.

Electromechanical Equivalence

Lorentz transformations in Minkowski space-time are isomorphic to (a subset of) symplectic transformations in two dimensions.

Assume we associate the RDM-coefficients of

- γ_0 with time-component of a 4-vector (i.e. energy $\mathcal{E}$)
- $\vec{\gamma}$ with the space-components of a 4-vector (i.e. momentum $\vec{P}$)
- γ_4, γ_5 and γ_6 with the electric field components $\vec{E}$
- γ_7, γ_8 and γ_9 with the magnetic field components $\vec{B}$

then we obtain a physically meaningful and transparent notation of the components of the force matrix and survey of symplectic transformations. I call this isomorphism **electromechanical equivalence**.

Symplectic Transformations I

The symplectic transformations generated by the ten RDMs $\gamma_0 \dots \gamma_9$ are

- γ_0: phase rotation.
- $\vec{\gamma}$ (γ_1, γ_2 and γ_3): phase boost along x, y, z.
- γ_4, γ_5 and γ_6: Lorentz boost along x, y, z.
- γ_7, γ_8 and γ_9: Spatial rotation about along x, y, z.

Rotations and Lorentz boosts are supposed to be well known. What do "phase rotations" and "phase boosts"?

The "phase rotation" $\mathbf{R}_0$ gives $\mathcal{E}' = \mathcal{E}$ and $\vec{B}' = \vec{B}$, but:

$$\begin{aligned} \vec{P}' &= \cos\varepsilon\,\vec{P} - \sin\varepsilon\,\vec{E} \\ \vec{E}' &= \cos\varepsilon\,\vec{E} + \sin\varepsilon\,\vec{P} \end{aligned}$$

The "phase boosts" $\mathbf{R}_k$ $k \in [1-3]$ are like Lorentz boosts, but with $\vec{E}$ and $\vec{P}$ exchanged. Hence they are identical to a sequence of 90° phase rotation + Lorentz boost + 90° inverse phase rotation.

Lorentz boosts: $\vec{E}\,\vec{B}$ and $\vec{E}^2 - \vec{B}^2$ are invariant. Phase boosts: $\vec{P}\,\vec{B}$ and $\vec{P}^2 - \vec{B}^2$ are invariant.

Envelope equation (from above):

$$\dot{\mathbf{S}} = \mathbf{F}\,\mathbf{S} - \mathbf{S}\,\mathbf{F}\,,$$

The Lorentz force equation then writes (using proper time τ):

$$\frac{d\mathbf{P}}{d\tau} = \dot{\mathbf{P}} = \frac{q}{2\,m}\,(\mathbf{F}\,\mathbf{P} - \mathbf{P}\,\mathbf{F})\;,$$

In the lab frame time $dt = \gamma\,d\tau$ this write as (setting $c = 1$):

$$\begin{aligned}
\frac{d\mathcal{E}}{d\tau} &= \frac{q}{m}\,\vec{P}\,\vec{E} & \qquad \frac{d\vec{P}}{d\tau} &= \frac{q}{m}\,\left(\mathcal{E}\,\vec{E} + \vec{P}\times\vec{B}\right)\\
\gamma\,\frac{d\mathcal{E}}{dt} &= q\,\gamma\,\vec{v}\,\vec{E} & \qquad \gamma\,\frac{d\vec{P}}{dt} &= \frac{q}{m}\,\left(m\,\gamma\,\vec{E} + m\,\gamma\,\vec{v}\times\vec{B}\right)\\
\frac{d\mathcal{E}}{dt} &= q\,\vec{v}\,\vec{E} & \qquad \frac{d\vec{P}}{dt} &= q\,\left(\vec{E} + \vec{v}\times\vec{B}\right)\;,
\end{aligned}$$

which are exactly the Lorentz force equations.

If one writes the 4-potential $\mathbf{\Phi}$ (4-current $\mathbf{J}$, 4-momentum $\mathbf{P}$) as a 4-vector using the RDMs $\gamma_0 \dots \gamma_3$ according to

$$\mathbf{\Phi} = \phi\,\gamma_0 + A_x\,\gamma_1 + A_y\,\gamma_2 + A_z\,\gamma_3 = \phi\,\gamma_0 + \vec{\gamma}\,\vec{A}\,,$$

the 4-derivative $\mathbf{D}$ as

$$\mathbf{D} = \partial_t\,\gamma_0 - \partial_x\,\gamma_1 - \partial_y\,\gamma_2 - \partial_z\,\gamma_3\,,$$

and the electromagnetic fields $\vec{E}$ and $\vec{B}$ as

$$\mathbf{F} = E_x\,\gamma_4 + E_y\,\gamma_5 + E_z\,\gamma_6 + B_x\,\gamma_7 + B_y\,\gamma_8 + B_z\,\gamma_9\,,$$

then the Maxwell equations can be written (remarkably compact) as:

$$\begin{aligned} \mathbf{F} &= -\mathbf{D}\,\mathbf{\Phi} \\ \mathbf{D}\,\mathbf{F} &= 4\,\pi\,\mathbf{J}\,, \end{aligned}$$

with the usual choice of units.

Symplectic Transformations III

The effect of a basic symplex γ_b is given by:

$$
\begin{aligned}
\mathbf{R}_b &= \exp\left(\gamma_b\,\varepsilon/2\right) \\
\mathbf{R}_b &= \begin{cases} \mathbf{1}\,\cos\left(\varepsilon/2\right) + \gamma_b\,\sin\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = -1 \\ \mathbf{1}\,\cosh\left(\varepsilon/2\right) + \gamma_b\,\sinh\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = 1 \end{cases} \\
\mathbf{R}_b^{-1} &= \exp\left(-\gamma_b\,\varepsilon/2\right) \\
&= \begin{cases} \mathbf{1}\,\cos\left(\varepsilon/2\right) - \gamma_b\,\sin\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = -1 \\ \mathbf{1}\,\cosh\left(\varepsilon/2\right) - \gamma_b\,\sinh\left(\varepsilon/2\right) & \text{for} \quad \gamma_b^2 = 1 \end{cases}
\end{aligned}
$$

Transformations with $\gamma_b^2 = -1$ are orthogonal transformations, i.e. *rotations* about angle ε, while those with $\gamma_b^2 = 1$ are *boosts* with “rapidity” ε.

Hence the transformed matrices are

$$\begin{aligned} \gamma_a' &= \mathbf{R}\,\gamma_a\,\mathbf{R} \\ &= \mathbf{R}\,\gamma_a\,\mathbf{R}^{-1} \\ &= (c - s\,\gamma_b)\,\gamma_a\,(c + s\,\gamma_b) \\ &= c^2\,\gamma_a - \gamma_b\,\gamma_a\,\gamma_b\,s^2 + c\,s\,(\gamma_a\,\gamma_b - \gamma_b\,\gamma_a)\,, \end{aligned}$$

where c and s are (hyperbolic) sine- and cosine-functions. The last term of the right vanishes if γ_b and γ_a commute, so that

$$\begin{aligned} \gamma_a' &= \gamma_a\,(c^2 - \gamma_b^2\,s^2) \\ &= \gamma_a \begin{cases} \cos^2(\varepsilon) + \sin^2(\varepsilon) = 1 & \text{for} \quad \gamma_b^2 = -1 \\ \cosh^2(\varepsilon) - \sinh^2(\varepsilon) = 1 & \text{for} \quad \gamma_b^2 = 1 \end{cases} \end{aligned}$$

(Unity transformation.)

If γ_b and γ_a anticommute, one finds:

$$\begin{aligned}\gamma_a' &= \gamma_a\,(c^2+\gamma_b^2\,s^2)+2\,c\,s\,\gamma_a\,\gamma_b\\ &= \begin{cases}\gamma_a\,\cos(2\,\varepsilon)+\gamma_a\,\gamma_b\,\sin(2\,\varepsilon) & \text{for}\quad \gamma_b^2=-1\\ \gamma_a\,\cosh(2\,\varepsilon)+\gamma_a\,\gamma_b\,\sinh(2\,\varepsilon) & \text{for}\quad \gamma_b^2=1\end{cases}\end{aligned}$$

Note: This treatment of Lorentz transformations and spatial rotations of the classical state vector $\psi=(q_1,p_1,q_2,p_2)^T$ is identical to the transformation formalism and properties of the Dirac spinor $\psi(\mathbf{x},t)$ in QED!

The most general force matrix **F** of a symplectic process can be written setting formally $\vec\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ according to the EMEQ as:

$$\begin{aligned}\mathbf{F} &= \mathcal{E}\,\gamma_0 + \vec P\,\vec\gamma + \vec E\,\gamma_0\,\vec\gamma + \vec B\,\gamma_{14}\,\gamma_0\,\vec\gamma \\ &= \begin{pmatrix} -E_x & E_z + B_y & E_y - B_z & B_x \\ E_z - B_y & E_x & -B_x & -E_y - B_z \\ E_y + B_z & B_x & E_x & E_z - B_y \\ -B_x & -E_y + B_z & E_z + B_y & -E_x \end{pmatrix} \\ &+ \begin{pmatrix} -P_z & \mathcal{E} - P_x & 0 & P_y \\ -\mathcal{E} - P_x & P_z & -P_y & 0 \\ 0 & P_y & -P_z & \mathcal{E} + P_x \\ P_y & 0 & -\mathcal{E} + P_x & P_z \end{pmatrix}.\end{aligned}$$

Bringing **F** to block-diagonal form then means to find a symplectic transformation **R** with $\mathbf{F}' = \mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1}$ such that $E'_y = B'_x = B'_z = P'_y = 0$.

The eigenvalues of the force matrix **F** are

$$
\begin{aligned}
\lambda &= \mathrm{Diag}(i\,\omega_1, -i\,\omega_1, i\,\omega_2, -i\,\omega_2) \\
K_1 &= \mathcal{E}^2 + \vec{B}^2 - \vec{E}^2 - \vec{P}^2 \\
K_2 &= -2\,\mathcal{E}\,\vec{P}\,(\vec{E}\times\vec{B}) + \mathcal{E}^2\,\vec{B}^2 + \vec{E}^2\,\vec{P}^2 \\
&- (\vec{E}\,\vec{P})^2 - (\vec{E}\,\vec{B})^2 - (\vec{P}\,\vec{B})^2 \\
\omega_1 &= \sqrt{K_1 + 2\,\sqrt{K_2}} \\
\omega_2 &= \sqrt{K_1 - 2\,\sqrt{K_2}} \\
\mathrm{Det}(\mathbf{F}) &= K_1^2 - 4\,K_2
\end{aligned}
$$

System stable $\Rightarrow$ eigenfrequencies are real:

$$
\begin{aligned}
K_2 &\geq 0 \\
K_1 &\geq 2\,\sqrt{K_2}
\end{aligned}
$$

"Fields" being zero:

$$\begin{aligned} K_1 &= \mathcal{E}^2 - \vec{P}^2 \equiv m^2 \\ K_2 &= 0 \\ \omega_1 &= \omega_2 = \sqrt{K_1} = m \\ \mathrm{Det}(\mathbf{F}) &= K_1^2 = m^4 \end{aligned}$$

"Energy + momentum" being zero:

$$\begin{aligned} \lambda &= \mathrm{Diag}(i\,\omega_1, -i\,\omega_1, i\,\omega_2, -i\,\omega_2) \\ K_1 &= \vec{B}^2 - \vec{E}^2 \\ K_2 &= -(\vec{E}\vec{B})^2 \end{aligned}$$

System stable $\Rightarrow$ eigenfrequencies must be real (only possible for $\vec{E}\vec{B} = 0$!!):

$$\omega_1 = \omega_2 = \sqrt{K_1} = \sqrt{\vec{B}^2 - \vec{E}^2}$$

Summary

- Algebraic extension of 1-dim. Courant-Snyder theory (CST) with Real Pauli Matrices (RPMs) is 2-dim. *CST* with Real Dirac Matrices (RDMs).
- The Ansatz using RDMs allow for a complete, final and comprehensive generalization as RDMs form a *complete basis* of all real 4×4-matrices.
- Based on the RDMs, a survey of symplectic transformations was given.
- The electromechanical equivalence (EMEQ) allows to use a familiar and meaningful nomenclature.
- A close and remarkable isomorphism of 2-dim. classical state vector and the Dirac spinor has been presented.

Referenzen

[1] V.I. Arnold: Mathematical Methods of Classical Mechanics; 2nd Ed., Springer, New York 2010.

[2] Andrej Wolski: Alternative Approach to General Coupled Linear Optics; PRST-AB 9, 024001 (2006).

[3] C. Baumgarten; Phys. Rev. ST Accel. Beams. 14, 114002 (2011); http://arxiv.org/abs/1109.2001.

[4] C. Baumgarten; Phys. Rev. ST Accel. Beams. 14, 114201 (2011); http://arxiv.org/abs/1109.1996.