1-Dim. Algebraic Courant-Snyder Theory: Real Pauli Matrices

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Motivation and Context

- Space charge dominated beam transport in cyclotrons yields "weird coupling".
- Computation of emittances and beam sizes requires decoupling.
- Teng/Edwards [1, 2] method does not work here.
- Extension of Teng/Edwards method presented at talk Dec. 2010.
- But: The extension was numerically "difficult" (a lot of "if-then-else"-statements required).
- No other publication found with a general, simple and working method.

Outline

This is a sequence of 3 talks:

- 1-Dim. Algebraic Courant-Snyder Theory (CST_1): Real Pauli Matrices.
- Algebraic CST_2: Real Dirac Matrices.
- Symplectic Decoupling in n Dimensions (CST_n).

Outline for today

- Time dependent harmonical oscillator (Hill's equation).
- Symplectic unit matrix and Hamilton equations of motion.
- Introduction of the "symplex" (Hamiltonian matrix).
- Symplectic condition for a linear transformation.
- Courant-Snyder theory with real Pauli matrices (RPMs).
- Transfer matrix + Floquet theoreme.
- Eigenvectors/Eigenvalues.
- Second moments (σ-matrix).
- Matching and Emittances.
- Symplectic transformations and "decoupling".

CST_1: Hill's Equation

Hill's Equation (time dependent harmonical oscialltor):

$$x''(s) = -k(s)\,x(s)\,.$$

Rewrite as linear differential equation:

$$\begin{aligned} q(s) &\equiv x(s) \\ p(s) &\equiv x'(s) \\ q'(s) &= p(s) \\ p'(s) &= -k(s)\,x(s) \end{aligned}$$

Or with $\psi = \left(\begin{array}{c} q \\ p \end{array}\right)$:

$$\psi' = \left(\begin{array}{cc} 0 & 1 \\ -k(s) & 0 \end{array}\right)\psi$$

Hamilton function with symmetric matrix **A**:

$$\begin{aligned}\mathcal{H} &= \omega\left(\frac{\gamma}{2}\,q^2+\alpha\,p\,q+\frac{\beta}{2}\,p^2\right)\\ &= \frac{\omega}{2}\begin{pmatrix} q\\ p\end{pmatrix}^T\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\begin{pmatrix} q\\ p\end{pmatrix}\\ &= \frac{1}{2}\,\psi^T\,\mathbf{A}\,\psi\,.\end{aligned}$$

Equations of Motion (EQOM):

$$\begin{aligned}\dot{q} &= \frac{\partial\mathcal{H}}{\partial p}=\omega\,(\alpha\,q+\beta\,p)\\ \dot{p} &= -\frac{\partial\mathcal{H}}{\partial q}=-\omega\,(\gamma\,q+\alpha\,p)\end{aligned}$$

CST_1: EQOM and Force Matrix

Introduce symplectic unit matrix η_0:

$$\eta_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

$\Rightarrow$ EQOM:

$$\begin{aligned} \dot\psi &= \omega \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \psi = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \psi \\ &= \eta_0\, \mathbf{A}\, \psi \\ &= \mathbf{F}\, \psi \end{aligned}$$

with force matrix $\mathbf{F}$ and Twiss-parameters α, β and γ.

Properties of Hamilton matrix $\mathbf{A}$:

- $\mathbf{A}$ is symmetric.
- $\mathbf{A}$ has dimension $n \times n$ and is twice the degrees of freedom N_f: $n = 2\,N_f$
- $\Rightarrow$ Number of parameters ν is $\nu = \frac{n\,(n+1)}{2}$.
- In 1-dim systems ($N_f = 1$): $\nu = 3$.
- In 2-dim systems ($N_f = 2$): $\nu = 10$.

Note: The same number of parameters is necessary to describe a σ-matrix for N_f degrees of freedom.

Properties of the force matrix $\mathbf{F} = \eta_0\,\mathbf{A}$:

- $\mathbf{F}^T = \eta_0\,\mathbf{F}\,\eta_0$ (Often called "Hamiltonian matrix" or "infinitesimal symplectic". I prefer "symplex").
- $\mathbf{F}$ is product of symmetric and antisymmetric matrix: $\Rightarrow\ \mathrm{Tr}(\mathbf{F}) = 0$.
- $\Rightarrow$ Sum of eigenvalues is zero.
- Superposition/Linearity: If $\mathbf{F}_1$ and $\mathbf{F}_2$ are symplices, then $\mathbf{F} = a_1\,\mathbf{F}_1 + a_2\,\mathbf{F}_2$ is a symplex.
- Multiplication Theorem: If $\mathbf{F}_1$ and $\mathbf{F}_2$ are force matrices, then $\mathbf{F} = \mathbf{F}_1\,\mathbf{F}_2$ is a force matrix, if (and only if) $\mathbf{F}_1$ and $\mathbf{F}_2$ anticommute. Proof:

$$\begin{array}{rcl}
(\mathbf{F}_1\,\mathbf{F}_2)^T & = & \mathbf{F}_2^T\,\mathbf{F}_1^T = \eta_0\,\mathbf{F}_2\,\eta_0\,\eta_0\,\mathbf{F}_1\,\eta_0 \\
 & = & -\eta_0\,\mathbf{F}_2\,\mathbf{F}_1\,\eta_0 = \eta_0\,(\mathbf{F}_1\,\mathbf{F}_2)\,\eta_0 \\
\Rightarrow & & -\mathbf{F}_2\,\mathbf{F}_1 = \mathbf{F}_1\,\mathbf{F}_2 \\
\Rightarrow & & \mathbf{F}_2\,\mathbf{F}_1 + \mathbf{F}_1\,\mathbf{F}_2 = 0
\end{array}$$

Given a (constant) linear transformation $\mathbf{R}$ of the state vector $\tilde{\psi} = \mathbf{R}\,\psi$, the transformation is **canonical**, if the Hamiltonian and the EQOM preserve their form (relativity principle):

$$\begin{aligned} \dot{\tilde{\psi}} &= \eta_0\,\nabla_{\tilde{\psi}}\,\tilde{\mathcal{H}} \\ \mathbf{R}\,\dot{\psi} &= \tilde{\mathbf{F}}\,\mathbf{R}\,\psi \\ \dot{\psi} &= \mathbf{R}^{-1}\,\tilde{\mathbf{F}}\,\mathbf{R}\,\psi \\ \Rightarrow \tilde{\mathbf{F}} &= \mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1} \end{aligned}$$

Preservation of form: $\mathbf{F}$ is symplex, hence $\tilde{\mathbf{F}}$ must be symplex:

$$\begin{aligned} (\mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1})^T &= (\mathbf{R}^{-1})^T\,\eta_0\,\mathbf{F}\,\eta_0\,\mathbf{R}^T \\ &= \eta_0\,\mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1}\,\eta_0 \\ (\mathbf{R}^{-1})^T\,\eta_0 &= \eta_0\,\mathbf{R} \\ \eta_0 &= \mathbf{R}^T\,\eta_0\,\mathbf{R} \end{aligned} \qquad \begin{aligned} \eta_0\,\mathbf{R}^T &= \mathbf{R}^{-1}\,\eta_0 \\ \mathbf{R}\,\eta_0\,\mathbf{R}^T &= \eta_0 \end{aligned}$$

i.e. $\mathbf{R}$ must be **symplectic**.

CST_1: Real Pauli Matrices (RPM)

All real 2×2 matrices can be written as a sum of the **real Pauli matrices (RPM)** η_0, η_1, $\eta_2 = \eta_0\,\eta_1$ and $\eta_3 = \mathbf{1}$:

$$\eta_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \eta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \eta_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note the following anticommutation relations:

$$\begin{aligned} \eta_0\,\eta_1 + \eta_1\,\eta_0 &= 0 \\ \eta_1\,\eta_2 + \eta_2\,\eta_1 &= 0 \\ \eta_0\,\eta_2 + \eta_2\,\eta_0 &= 0\,. \end{aligned}$$

The squares are: $\eta_1^2 = \eta_2^2 = -\eta_0^2 = \mathbf{1}$.

CST_1: RPM II

Any real-valued 2×2 matrix $\mathbf{M}$ can be written as

$$\mathbf{M} = \sum_{k=0}^{3} m_k\,\eta_k\,,$$

where the RPM-coefficients m_k are given by the "scalar product":

$$m_k = \mathrm{sign}(\eta_k)\;Tr(\mathbf{M}\,\eta_k + \eta_k\,\mathbf{M})/4\,.$$

The signature $\mathrm{sign}(\eta_k)$ of a RPM is given by

$$\mathrm{sign}(\eta_k) = Tr(\eta_k^2)/2\,.$$

Note: The antisymmetric RPM η_0 has a negative signature, the symmetric RPMs a positive.

CST_1: RPM III

Recall a matrix $\mathbf{F}$ that fulfills

$$\mathbf{F}^T = \eta_0\,\mathbf{F}\,\eta_0$$

is a **symplex**. The basis matrices η_0, η_1 and η_2 are symplices. Superposition principle $\Rightarrow$ general $\mathbf{F}$:

$$\Rightarrow \mathbf{F} = \omega\,\left(\frac{\beta+\gamma}{2}\,\eta_0 + \frac{\beta-\gamma}{2}\,\eta_1 + \alpha\,\eta_2\right)\,.$$

Note the square of $\mathbf{F}$:

$$\begin{array}{rcl} \mathbf{F}^2/\omega^2 & = & (\alpha^2 - \beta\,\gamma)\,\mathbf{1} \\ \Rightarrow & & \mathbf{F}^2 = -\omega^2\,\mathbf{1} \end{array}$$

for normalization $\beta\,\gamma - \alpha^2 = 1$.

CST_1: Transfer Matrix

The time evolution is a canonical (i.e. symplectic) transformation. $\Rightarrow$ the solution $\psi(s)$ can be written as

$$\begin{aligned} \psi(s) &= \mathbf{M}(s)\,\psi(0) \\ \dot\psi(s) &= \dot{\mathbf{M}}(s)\,\psi(0) \\ &= \dot{\mathbf{M}}\,\mathbf{M}^{-1}\,\mathbf{M}\psi(0) \\ &= \dot{\mathbf{M}}\,\mathbf{M}^{-1}\,\psi(s) \\ &= \mathbf{F}\,\psi(s) \end{aligned}$$

with a symplectic "transfer matrix" $\mathbf{M}$, so that:

$$\begin{aligned} \mathbf{F} &= \dot{\mathbf{M}}\,\mathbf{M}^{-1} \\ \dot{\mathbf{M}} &= \mathbf{F}\,\mathbf{M} \end{aligned}$$

Compare to $\dot\psi = \mathbf{F}\,\psi \Rightarrow$ The columns of $\mathbf{M}$ are solutions of the EQOM.

EQOM:

$$\dot{\psi} = \mathbf{F}\,\psi\,.$$

Floquet-Theorem: If $\mathbf{F}$ is periodic $\mathbf{F}(s+L) = \mathbf{F}(s)$, then the transfer matrix $\mathbf{M}$ can be written in the form:

$$\mathbf{M}(s) = \mathbf{K}(s)\,\exp\left(\bar{\mathbf{F}}\,s\right),$$

where $\mathbf{K}(s)$ is symplectic and periodic.

$$\begin{array}{rclcrcl} \mathbf{M}(0) & = & \mathbf{1} & \Rightarrow & \mathbf{K}(0) & = & \mathbf{1} \\ \mathbf{K}(s+L) & = & \mathbf{K}(s) & \Rightarrow & \mathbf{K}(L) & = & \mathbf{1} \end{array}$$

Transfer matrix of one period of length L:

$$\begin{array}{rcl} \mathbf{M}(L) = \mathbf{M}_L & = & \mathbf{K}(L)\,\exp\left(\bar{\mathbf{F}}\,L\right) \\ & = & \exp\left(\bar{\mathbf{F}}\,L\right), \end{array}$$

CST_1: Floquet Theorem II

Consider a constant force matrix. EQOM:

$$\dot{\psi} = \bar{\mathbf{F}}\,\psi\,.$$

Solution:

$$\psi(s) = \exp\left(\bar{\mathbf{F}}\,s\right)\psi(0)\,.$$

$\Rightarrow$ $\bar{\mathbf{F}}$ is the "average" force matrix [5]:

$$\bar{\mathbf{F}} = \frac{1}{L}\ln\left(\mathbf{M}_L\right),$$

and the logarithm of the symplectic one-turn-transfer matrix $\mathbf{M}_L$.

$$
\begin{aligned}
\psi(s) &= \mathbf{K}(s)\,\exp(\bar{\mathbf{F}}\,s)\,\psi(0) \\
\mathbf{K}^{-1}(s)\,\psi(s) &= \exp(\bar{\mathbf{F}}\,s)\,\mathbf{K}(0)\,\mathbf{K}^{-1}(0)\psi(0)
\end{aligned}
$$

Since $\mathbf{K}$ is symplectic, the transformation

$$
\begin{aligned}
\tilde{\psi}(s) &= \mathbf{K}^{-1}(s)\,\psi(s) \\
\tilde{\psi}(s) &= \exp(\bar{\mathbf{F}}\,s)\,\mathbf{K}(0)\,\tilde{\psi}(0)
\end{aligned}
$$

is canonical. For a given fixed reference position s_0 we have $\mathbf{K}(s_0) = \mathbf{1}$. Let $s_0 = 0$:

$$
\begin{aligned}
\tilde{\psi}(s) &= \exp(\bar{\mathbf{F}}\,s)\,\tilde{\psi}(0) \\
\tilde{\mathbf{M}}(s) &= \exp(\bar{\mathbf{F}}\,s)
\end{aligned}
$$

The beam optics of a linear periodic system can be equivalently described by a system with constant $\bar{\mathbf{F}}$.

From

$$\begin{aligned} \mathbf{M}(s) &= \exp(\bar{\mathbf{F}}\,s) \\ \bar{\mathbf{F}}^2 &= -\omega^2\,\mathbf{1} \end{aligned}$$

it follows that

$$\begin{aligned} \mathbf{M}(s) &= \exp(\bar{\mathbf{F}}\,s) = \sum_{k=0}^{\infty} \frac{(\bar{\mathbf{F}}\,s)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(\bar{\mathbf{F}}\,s)^{(2k)}}{2k!} + \sum_{k=0}^{\infty} \frac{(\bar{\mathbf{F}}\,s)^{(2k+1)}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-)^k \frac{(\omega\,s)^{(2k)}}{2k!} + \bar{\mathbf{F}}/\omega \sum_{k=0}^{\infty} (-)^k \frac{(\omega\,s)^{(2k+1)}}{(2k+1)!} \\ &= \mathbf{1}\,\cos(\omega\,s) + \sin(\omega\,s)\,\bar{\mathbf{F}}/\omega \end{aligned}$$

From

$$\bar{\mathbf{F}}/\omega = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} .$$

one finds:

$$\mathbf{M}(s) = \mathbf{1}\,\cos(\omega\,s) + \sin(\omega\,s) \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} .$$

with

$$\mathbf{M}^N(s) = \mathbf{1}\,\cos(N\,\omega\,s) + \sin(N\,\omega\,s) \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} .$$

CST_1: Eigensystems Ia

- The force matrix **F** is a product of the **antisymmetric** matrix η_0 and the **symmetric** matrix **A**.

$\Rightarrow$ The trace of **F** is zero.

$\Rightarrow$ The sum of the eigenvalues is zero.

- **Stable** systems have only imaginary eigenvalues [4].

$\Rightarrow$ The eigenvalues are a pair $(i\,\omega, -i\,\omega)$.

Eigensystem of force matrix:

$$\begin{aligned}
\bar{\mathbf{F}} &= \mathbf{E}\,\lambda\,\mathbf{E}^{-1} \\
\bar{\mathbf{F}}^n &= \mathbf{E}\,\lambda^n\,\mathbf{E}^{-1} \\
\Rightarrow &\quad f(\bar{\mathbf{F}}) = \mathbf{E}\,f(\lambda)\,\mathbf{E}^{-1} \\
\Rightarrow &\quad \exp(\bar{\mathbf{F}}\,s) = \mathbf{E}\,e^{\lambda\,s}\,\mathbf{E}^{-1}
\end{aligned}$$

$\Rightarrow$ transfer matrix:

$$\begin{aligned}
\tilde{\mathbf{M}}(s) &= \mathbf{E}\,\exp(\lambda\,s)\,\mathbf{E}^{-1} \\
&= \mathbf{E}\,\Lambda\,\mathbf{E}^{-1}
\end{aligned}$$

with

$$\begin{aligned}
\lambda &= \mathrm{Diag}(-i\,\omega, i\,\omega) = -i\,\omega\,\eta_2 \\
\Lambda &= \exp(\lambda\,s) = \mathrm{Diag}(e^{-i\,\omega\,s}, e^{i\,\omega\,s}) \\
&= \mathbf{1}\,\cos(\omega\,s) - i\,\eta_2\,\sin(\omega\,s)
\end{aligned}$$

It follows that:

$$\begin{aligned}\bar{\mathbf{F}} &= \mathbf{E}\,\lambda\,\mathbf{E}^{-1}\\ &= -i\,\omega\,\mathbf{E}\,\eta_2\,\mathbf{E}^{-1}\\ \tilde{\mathbf{M}}(s) &= \mathbf{E}\,\Lambda\,\mathbf{E}^{-1}\\ &= \mathbf{1}\,\cos(\omega\,s) - i\,\mathbf{E}\,\eta_2\,\mathbf{E}^{-1}\,\sin(\omega\,s)\end{aligned}$$

where $\mathbf{E}$ is given by (with $\beta\,\gamma-\alpha^2=1$)

$$\begin{aligned}\mathbf{E} &= \frac{1}{\sqrt{2\,\gamma}}\begin{pmatrix} i-\alpha & -i+\alpha \\ \gamma & \gamma \end{pmatrix}\\ \mathbf{E}^{-1} &= \frac{1}{\sqrt{2\,\gamma}}\begin{pmatrix} -i\,\gamma & 1-i\,\alpha \\ i\,\gamma & 1+i\,\alpha \end{pmatrix}.\end{aligned}$$

If u_1 and u_2 are the eigenvectors (columns of $\mathbf{E}$) of the force matrix, then:

$$\dot{u}_i = \bar{\mathbf{F}}\,u_i = \pm\,i\,\omega\,u_i\,,$$

and hence:

$$u_i(s) = e^{\pm\,i\,\omega\,s}\,u_i(0)\,.$$

Furthermore one finds with the definition $\bar{u}_i = u_i^\dagger\,\eta_0$:

$$\begin{aligned}\bar{u}_1\,u_1 &= -i & \qquad \bar{u}_2\,u_2 &= i\\ \bar{u}_1\,u_2 &= 0 & \qquad \bar{u}_2\,u_1 &= 0\end{aligned}$$

(Orthogonality equations)

$$\begin{aligned}\mathbf{1} &= i\,(u_1\,\bar{u}_1 - u_2\,\bar{u}_2)\\ \bar{\mathbf{F}} &= \omega\,(u_1\,\bar{u}_1 + u_2\,\bar{u}_2)\\ \bar{\mathbf{F}}\,u_1 &= -i\,\omega\,u_1\\ \bar{\mathbf{F}}\,u_2 &= i\,\omega\,u_2\end{aligned}$$

What is the advantage of the eigensystem?

- The eigensystem is an especially simple way to express the same physical situation.
- The EQOM are then eigenvalue-equations.
- The transformation into the eigensystem is not symplectic.
- $\Rightarrow$ The equations of motion are modified.
- The use of eigensystems is paid by the use of complex numbers.
- Knowledge of eigenvectors allows to compute matched beam ellipsoid (below).

Definition of matrix of second moments:

$$\sigma = \langle \psi \, \psi^T \rangle \,.$$

Time evolution:

$$\begin{aligned}
\dot\sigma &= \langle \dot\psi \, \psi^T \rangle + \langle \psi \, \dot\psi^T \rangle \\
&= \langle \mathbf{F} \, \psi \, \psi^T \rangle + \langle \psi \, \psi^T \, \mathbf{F}^T \rangle \\
&= \mathbf{F} \, \sigma + \sigma \, \mathbf{F}^T \\
&= \mathbf{F} \, \sigma + \sigma \, \eta_0 \, \mathbf{F} \, \eta_0 \\
\dot\sigma \, \eta_0 &= \mathbf{F} \, \sigma \, \eta_0 + \sigma \, \eta_0 \, \mathbf{F} \, \eta_0^2 \\
&= \mathbf{F} \, (\sigma \, \eta_0) - (\sigma \, \eta_0) \, \mathbf{F}
\end{aligned}$$

Define matrix $\mathbf{S} = \sigma \, \eta_0 = \langle \psi \psi^T \, \eta_0 \rangle = \langle \psi \, \bar\psi \rangle$ with

$$\dot{\mathbf{S}} = \mathbf{F} \, \mathbf{S} - \mathbf{S} \, \mathbf{F} \,.$$

CST_1: **S**-Matrix

The **S**-matrix is a constant of motion, if

$$\begin{aligned}
\dot{\mathbf{S}} &= \mathbf{F}\,\mathbf{S} - \mathbf{S}\,\mathbf{F} = 0 \\
0 &= \mathbf{E}\,\lambda\,\mathbf{E}^{-1}\,\mathbf{S} - \mathbf{S}\,\mathbf{E}\,\lambda\,\mathbf{E}^{-1} \\
&= \lambda\,\mathbf{E}^{-1}\,\mathbf{S}\,\mathbf{E} - \mathbf{E}^{-1}\,\mathbf{S}\,\mathbf{E}\,\lambda \\
&= \lambda\,\mathbf{D} - \mathbf{D}\,\lambda\,.
\end{aligned}$$

Since only diagonal matrices commute with diagonal matrices, **D** must be diagonal, so that the force matrix and the **S**-matrix share the same eigenvectors:

$$\mathbf{S} = \mathbf{E}\,\mathbf{D}\,\mathbf{E}^{-1}\,.$$

Generally: Commuting matrices share the same system of eigenvectors.

CST_1: Matching

A beam is matched, if the σ-matrix is unchanged after one turn. If **M** is one-turn transfer matrix, then

$$\begin{aligned} \sigma &= \mathbf{M}\,\sigma\,\mathbf{M}^T \\ \sigma\,\eta_0 &= \mathbf{M}\,\sigma\,\mathbf{M}^T\,\eta_0 \\ \mathbf{S} &= -\mathbf{M}\,\mathbf{S}\,\eta_0\,\mathbf{M}^T\,\eta_0 \end{aligned}$$

M is symplectic, i.e.:

$$\begin{aligned} \mathbf{M}^T\,\eta_0\,\mathbf{M} &= \eta_0 \\ -\eta_0\,\mathbf{M}^T\,\eta_0 &= \mathbf{M}^{-1}\,, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{S} &= \mathbf{M}\,\mathbf{S}\,\mathbf{M}^{-1} \\ \mathbf{S}\,\mathbf{M} &= \mathbf{M}\,\mathbf{S} \\ 0 &= \mathbf{S}\,\mathbf{M} - \mathbf{M}\,\mathbf{S} \end{aligned}$$

Matching $\Rightarrow$ **M** and **S** share the same system of eigenvectors [6].

The relevant matrices in stable (focusing) systems:

$$\begin{aligned} \mathbf{F} &= \mathbf{E}\,\lambda\,\mathbf{E}^{-1} = \mathbf{E}\,\mathrm{Diag}(i\,\omega, -i\omega)\,\mathbf{E}^{-1} \\ \mathbf{M} &= \mathbf{E}\,\Lambda\,\mathbf{E}^{-1} = \mathbf{E}\,\mathrm{Diag}(e^{i\,\omega\,s}, e^{-i\,\omega\,s})\,\mathbf{E}^{-1} \\ \mathbf{S} &= \mathbf{E}\,\mathbf{D}\,\mathbf{E}^{-1} = \mathbf{E}\,\mathrm{Diag}(-i\,\varepsilon, i\varepsilon)\,\mathbf{E}^{-1} \end{aligned}$$

Eigenvalues of $\mathbf{S} = \sigma\,\eta_0$ are $\pm$ emittance ε [6].
Possible method to obtain matched beam ellipsoid from the one-turn (one-sector) transfer matrix **M or** the force matrix **F**:

- Compute matrix of eigenvectors **E**.
- Compose $\mathbf{S} = \mathbf{E}\,\mathbf{D}\,\mathbf{E}^{-1}$.
- Obtain beam sizes by $\sigma = \mathbf{S}\,\eta_0 = -\mathbf{E}\,\mathbf{D}\,\mathbf{E}^{-1}\,\eta_0$.
- But: Decoupling transformations must be symplectic (emittance preservation)!

But according to F. Hinterberger [7] the matched beam ellipsoid is given by:

$$\sigma = \varepsilon \left(\begin{array}{cc} \beta & -\alpha \\ -\alpha & \gamma \end{array} \right) .$$

So why to compute eigenvectors at all?

- 1-dim.: Eigenvectors not required to obtain matched beam.
- 2-dim.: Eigenvectors not required: **decoupling** to block-diagonal form sufficient.
- n-dim.: Iterative **decoupling** to block-diagonal form sufficient.
- Block-diagonal form $\Rightarrow$ use 1-Dim. Courant Snyder Theory.

Symplectic transfer matrix $\mathbf{M} = \exp(\mathbf{F}\,s)$ is exponential of symplex $\mathbf{F}$.
Any exponential of a symplex $\mathbf{F} = \eta_0\,F^T\,\eta_0$ is symplectic:

$$
\begin{aligned}
\mathbf{M}\,\eta_0\,\mathbf{M}^T &= \left(\sum_{k=0}^{\infty}\frac{\mathbf{F}^k}{k!}\right)\,\eta_0\,\left(\sum_{k=0}^{\infty}\frac{(\mathbf{F}^T)^k}{k!}\right)\\
&= \left(\sum_{k=0}^{\infty}\frac{\mathbf{F}^k}{k!}\right)\,\eta_0\,\left(\sum_{k=0}^{\infty}(-)^{k-1}\eta_0\,\frac{\mathbf{F}^k}{k!}\,\eta_0\right)\\
&= \left(\sum_{k=0}^{\infty}\frac{\mathbf{F}^k}{k!}\right)\,\left(\sum_{k=0}^{\infty}\frac{(-\mathbf{F})^k}{k!}\,\eta_0\right)\\
&= \exp(\mathbf{F})\,\exp(-\mathbf{F})\,\eta_0\\
&= \eta_0
\end{aligned}
$$

There are 3 basic symplices in 1-dim. systems: η_0, η_1, η_2 and hence 3 independent symplectic transformations.

The exponentials of the basic matrices are:

$$\begin{aligned} \mathbf{R}_0 &= \exp(\eta_0\,\varepsilon_0/2) = \cos(\varepsilon_0/2)\,\mathbf{1} + \sin(\varepsilon_0/2)\,\eta_0 \\ \mathbf{R}_1 &= \exp(\eta_1\,\varepsilon_1/2) = \cosh(\varepsilon_1/2)\,\mathbf{1} + \sinh(\varepsilon_1/2)\,\eta_1 \\ \mathbf{R}_2 &= \exp(\eta_2\,\varepsilon_2/2) = \cosh(\varepsilon_2/2)\,\mathbf{1} + \sinh(\varepsilon_2/2)\,\eta_2 \end{aligned}$$

$\mathbf{R}_0$ is a "phase rotation", i.e. a rotation in phase space:

$$\begin{aligned} \mathbf{R}_0\,\eta_0\,\mathbf{R}_0^{-1} &= \eta_0 \\ \mathbf{R}_0\,\eta_1\,\mathbf{R}_0^{-1} &= \cos(\varepsilon_0)\,\eta_1 + \sin(\varepsilon_0)\,\eta_2 \\ \mathbf{R}_0\,\eta_2\,\mathbf{R}_0^{-1} &= \cos(\varepsilon_0)\,\eta_2 - \sin(\varepsilon_0)\,\eta_1 \end{aligned}$$

$\mathbf{R}_1$ and $\mathbf{R}_2$ are called "phase boosts" due to their formal similarity with Lorentz boosts.
The phase boosts with η_1:

$$\begin{aligned}
\mathbf{R}_1\,\eta_0\,\mathbf{R}_1^{-1} &= \cosh(\varepsilon_1)\,\eta_0 - \sinh(\varepsilon_1)\,\eta_2 \\
\mathbf{R}_1\,\eta_1\,\mathbf{R}_1^{-1} &= \eta_1 \\
\mathbf{R}_1\,\eta_2\,\mathbf{R}_1^{-1} &= \cosh(\varepsilon_1)\,\eta_2 - \sinh(\varepsilon_1)\,\eta_0
\end{aligned}$$

The phase boosts with η_2:

$$\begin{aligned}
\mathbf{R}_2\,\eta_0\,\mathbf{R}_2^{-1} &= \cosh(\varepsilon_2)\,\eta_0 + \sinh(\varepsilon_2)\,\eta_1 \\
\mathbf{R}_2\,\eta_1\,\mathbf{R}_2^{-1} &= \cosh(\varepsilon_2)\,\eta_1 + \sinh(\varepsilon_2)\,\eta_0 \\
\mathbf{R}_2\,\eta_2\,\mathbf{R}_2^{-1} &= \eta_2
\end{aligned}$$

CST_1: "Decoupling"-Transformation

The symplectic transformations can be used (also in combination) to bring **F** to the normal form. For instance, if we chose

$$\begin{aligned}
\mathbf{U} &= \exp\left(x\,\eta_0 - x\,\eta_1 + y\,\eta_2\right) = \begin{pmatrix} e^y & 0 \\ -\frac{2\,x}{y}\,\sinh(y) & e^{-y} \end{pmatrix} \\
x &= \frac{\alpha\,\log(\sqrt{\beta})}{\beta-1} \\
y &= \log(\sqrt{\beta})\,,
\end{aligned}$$

then the calculation yields [3]:

$$\begin{aligned}
\mathbf{U} &= \frac{1}{\sqrt{\beta}}\begin{pmatrix} \beta & 0 \\ -\alpha & 1 \end{pmatrix} \\
\mathbf{U}^{-1} &= \frac{1}{\sqrt{\beta}}\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}
\end{aligned}$$

The transformation results in:

$$\begin{aligned}
\tilde{\mathbf{F}} &= \mathbf{U}\,\mathbf{F}\,\mathbf{U}^{-1} = \omega\,\mathbf{U}\left(\begin{array}{cc} \alpha & \beta \\ -\gamma & -\alpha \end{array}\right)\mathbf{U}^{-1} = \omega\,\eta_0 \\
\tilde{\mathbf{F}} &= \eta_0\,\tilde{\mathbf{A}} \\
\Rightarrow &\qquad \tilde{\mathbf{A}} = \omega\,\mathbf{1} \\
\tilde{\mathcal{H}} &= \frac{\omega}{2}\,\tilde{\psi}^T\,\tilde{\psi} \\
&= \frac{\omega}{2}\,(\tilde{q}^2 + \tilde{p}^2)
\end{aligned}$$

The transformation is the so-called "Floquet-Transformation" [7]. It transforms the Hamiltonian to what we call "normal form".

CST_1: "Decoupling"-Transformation III

Note that the Floquet-Transformation is only determined up to an orthogonal symplectic transformation $\mathbf{V}$:

$$\tilde{\mathbf{F}} = \mathbf{U}\,\mathbf{F}\,\mathbf{U}^{-1} = \omega\,\mathbf{U}\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\mathbf{U}^{-1} = \omega\,\eta_0$$

$$\tilde{\mathbf{F}} = \mathbf{V}\,\mathbf{U}\,\mathbf{F}\,\mathbf{U}^{-1}\,\mathbf{V}^{-1} = \omega\,\mathbf{V}\,\eta_0\,\mathbf{V}^T = \omega\,\eta_0$$

The only orthogonal symplectic transformation (in 1-dim. systems) is the phase rotation:

$$\mathbf{V} = \exp\left(\eta_0\,\phi\right).$$

Hence the Floquet-Transformation has the *general form*

$$\mathbf{U} = \exp\left(\eta_0\,\phi\right)\,\exp\left(x\,\eta_0 - x\,\eta_1 + y\,\eta_2\right).$$

Summary

- Linearized system with N_f degrees of freedom.
- Linear equations of motion for $2\,N_f$ variables.
- Use Floquet theory to get rid of time dependence.
- Hamiltonian with symmetric $2\,N_f \times 2\,N_f$-matrix, i.e. $\nu = \frac{2\,N_f\,(2\,N_f+1)}{2}$ independent components.
- $\Rightarrow \nu$ linear independent basic "symplices".
- $\Rightarrow \nu$ second moments including all correlations.
- $\Rightarrow \nu$ symplectic transformations.
- $N_f = 1$ yields $\nu = 3$: Usual Courant-Snyder theory, relatively simple. Not really worth the formalism.
- $N_f = 2$ yields $\nu = 10$: Generalized Courant-Snyder theory with real Dirac matrices. Not trivial. Next talk.
- $N_f > 2$: Use "Jacobi" method for decoupling: Talk in 2 weeks.

CST_1: Summary (cont.)

- CST_1 can algebraically be expressed by the *real Pauli matrices*.
- The eigensystems of the (periodic) transfer matrix $\mathbf{M}$, the (average) force matrix $\mathbf{F}$ and of the $\mathbf{S} = \sigma\,\eta_0$-matrix are identical.
- The Floquet-transformation is symplectic (hence *canonical*).

Thank you for your attention.

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